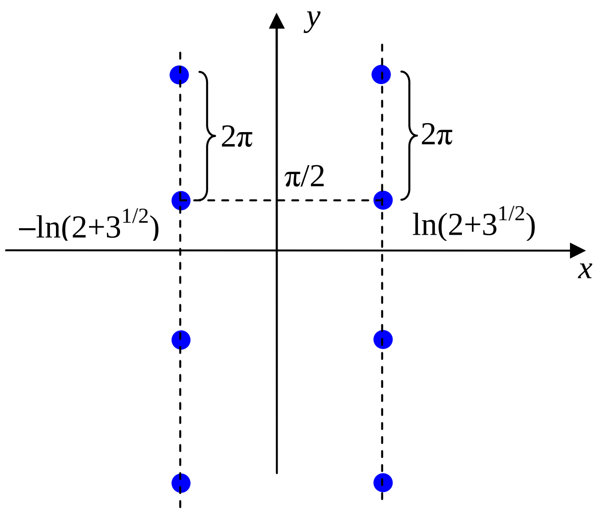
Math 656 • March 10, 2011
Midterm Examination Solutions

1) (14pts) Derive the expression for $\sinh^{-1} z$ (arcsinh z) using the definition of sinh w in terms of exponentials, and use it to find **all** values of $\sinh^{-1}(2i)$. Plot these values as points in the complex plane. Make sure your points agree with the period of the hyperbolic sine function.

$$z = \sinh w = \frac{e^w - e^{-w}}{2} \Rightarrow e^w - e^{-w} - 2z = 0 \,\Big|\times e^z \Rightarrow \underbrace{e^{2w}}_{s^2} - 2w\underbrace{e^w}_{s} - 1 = 0$$

$$s^2 - 2zs - 1 = 0 \Rightarrow s = z + (z^2+1)^{1/2} \Rightarrow \boxed{w = \sinh^{-1} z = \log\left\{z + (z^2+1)^{1/2}\right\}}$$

$$\Rightarrow \sinh^{-1}(2i) = \log\left\{2i + (-4+1)^{1/2}\right\} = \log\left\{2i \pm i\sqrt{3}\right\} = \ln(2\pm\sqrt{3}) + i\frac{\pi}{2} + i\,2\pi k = \pm\ln(2+\sqrt{3}) + i\frac{\pi}{2} + i\,2\pi k$$



This agrees with the period of hyperbolic sine function, which is $2\pi i$

2) (15pts) Show that images of vertical lines under transformation $w = \cos z$ are hyperbolic $\left(\frac{u^2}{a^2} - \frac{v^2}{b^2} = \pm 1\right)$. What shape are images of horizontal lines? What is the angle between these two sets of curves?

$$w = \cos z = \cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \boxed{\cos x \cosh y - i \sin x \sinh y}$$

1. Image of vertical lines: $z = c + iy,\ -\infty < y < \infty,\ c$ is fixed

$$w = \underbrace{\cos c\ \cosh y}_{u} - i\underbrace{\sin c\ \sinh y}_{-v}$$

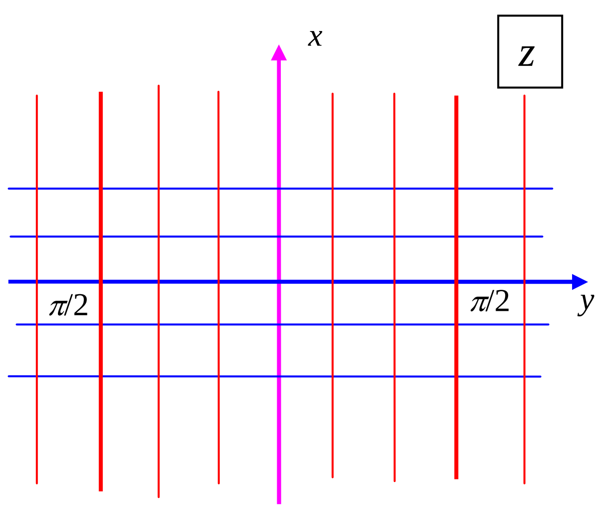
Use identity $\cosh^2 y - \sinh^2 y = 1$: $\boxed{\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 1}$ A hyperbola (unless $\cos c = 0$ or $\sin c = 0$)

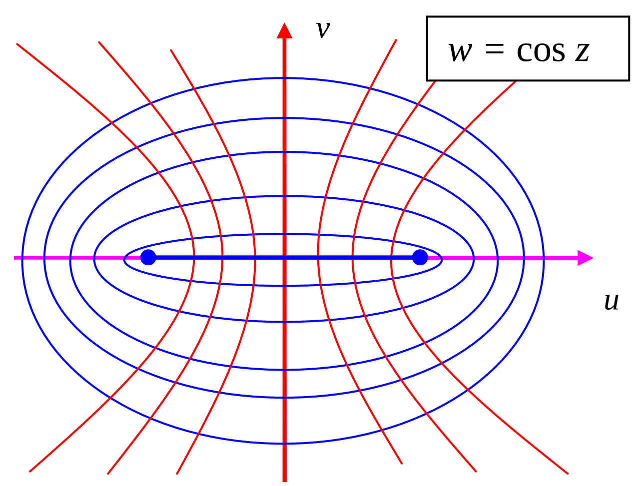
If $\cos c = 0,$ the image is the v-axis (see figure). When $\sin c = 0,$ the images are real: $(-\infty, -1)$ or $(1, +\infty)$

2. Image of horizontal lines: $z = x + ic,\ -\infty < x < \infty,\ c$ is fixed

$$w = \underbrace{\cos x\ \cosh c}_{u} - i\underbrace{\sin x\ \sinh c}_{-v}$$

Use identity $\cos^2 x + \sin^2 x = 1$: $\boxed{\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1}$ An ellipse (or a line [-1,1] if sinh c = 0)





This is a conformal mapping, which preserves angles, so images of horizontal and vertical lines are orthogonal wherever $(\cos z)' \neq 0$, since the horizontal and vertical lines are orthogonal.

3. (16pts) Is the function $f(z) = z/\bar{z}$ continuous for all z? Is it differentiable anywhere? Is it analytic anywhere? Prove your answers directly (using limits), and verify your answer about analyticity using Cauchy-Riemann equations.

$f(z)$ is continuous wherever its real and imaginary parts are continuous

$$f(z) = \frac{z}{\bar{z}} = \frac{z^2}{|z|^2} = \underbrace{\frac{x^2 - y^2}{x^2 + y^2}}_{u} + i\underbrace{\frac{2xy}{x^2 + y^2}}_{v} \qquad \text{(or, in polar form } f(z) = re^{i\theta}/re^{-i\theta} = e^{2i\theta} = \underbrace{\cos(2\theta)}_{u} + i\,\underbrace{\sin(2\theta)}_{v}\text{)}$$

$\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous apart from point $x = y = 0$, where $f(z)$ is not even defined, so the only discontinuity is $z = 0$

Now let's examine differentiability:

The simplest way to show non-analyticity in this case is to differentiate in polar coordinates (see homework #4), but we can also use the usual Cartesian representation:

$$\frac{df}{dz} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0}\left\{\frac{1}{h}\left(\frac{z+h}{\bar{z}+\bar{h}} - \frac{z}{\bar{z}}\right)\right\} = \lim_{h\to 0}\left\{\frac{\bar{z}(z+h) - z(\bar{z}+\bar{h})}{h\bar{z}(\bar{z}+\bar{h})}\right\} = \lim_{h\to 0}\left\{\frac{\bar{z}h - z\bar{h}}{h\bar{z}(\bar{z}+\bar{h})}\right\}$$

Let's take the derivative along the horizontal directions: $h = \bar{h} = \Delta x \to 0$

$$\frac{df}{dz} = \lim_{\Delta x\to 0}\left\{\frac{\Delta x}{\Delta x}\frac{\bar{z} - z}{\bar{z}(\bar{z}+\Delta x)}\right\} = \frac{\bar{z} - z}{(\bar{z})^2} = \boxed{-2i\frac{\operatorname{Im} z}{(\bar{z})^2}}$$

Now let's differentiate along the vertical direction: $h = i\Delta y,\ \bar{h} = -i\Delta y,\ \Delta y \to 0$

$$\frac{df}{dz} = \lim_{\Delta y\to 0}\left\{\frac{i\,\Delta y}{i\,\Delta y}\frac{\bar{z} + z}{\bar{z}(\bar{z} - i\Delta y)}\right\} = \frac{\bar{z} + z}{(\bar{z})^2} = \boxed{2\frac{\operatorname{Re} z}{(\bar{z})^2}}$$

These two expressions are never equal (note that z=0 is outside of domain of definition)

Also, let's check Cauchy-Riemann equations:

$$\left.\begin{aligned} u &= \frac{x^2 - y^2}{x^2 + y^2} \Rightarrow u_x = \frac{2x(x^2 + y^2) - 2x(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2} \\ v &= \frac{2xy}{x^2 + y^2} \Rightarrow v_y = \frac{2x(x^2 + y^2) - 2y\cdot 2xy}{(x^2 + y^2)^2} = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^2} \end{aligned}\right\} \text{Equal when } 3y^2 = x^2 \text{ or } x = 0$$

$$\left.\begin{aligned} u &= \frac{x^2 - y^2}{x^2 + y^2} \Rightarrow u_y = \frac{-2y(x^2 + y^2) - 2y(x^2 - y^2)}{(x^2 + y^2)^2} = -\frac{4yx^2}{(x^2 + y^2)^2} \\ v &= \frac{2xy}{x^2 + y^2} \Rightarrow -v_x = -\frac{2y(x^2 + y^2) - 2x\cdot 2xy}{(x^2 + y^2)^2} = \frac{2y(x^2 - y^2)}{(x^2 + y^2)^2} \end{aligned}\right\} \text{Equal when } 3x^2 = y^2 \text{ or } y = 0$$

$\Rightarrow$ Both Cauchy-Riemann equations cannot be satisfied

Note: it is easier to use polar C-R equations: $u_r = v_\theta/r$, $v_r = -u_\theta/r$ (here $u=\cos(2\theta)$, $v=\sin(2\theta)$, so $u_r = v_r = 0$, while $u_\theta = -2\sin(2\theta)$ and $v_\theta = 2\cos(2\theta)$, and the polar C-R equations are not satisfied)

4) (16pts) Find *all* branch points of $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$, and find the branch cut for *any* branch choice for this function (hint: this is simpler than examples we solved in class). How much does the function jump across the branch cut(s)?

Since $\frac{1}{2}\log\frac{1+z}{1-z} = \frac{1}{2}\left[\log(1+z) - \log(1-z)\right]$ it is clear that the branch points are $z = -1$ and $z = 1$

(note though that above logarithmic identity will not hold for all branch choices)

Note that $z=\infty$ is not a branch point since plugging in $z=1/t$ yields $\frac{1}{2}\log\frac{1+1/t}{1-1/t} = \frac{1}{2}\log\frac{t+1}{t-1}$

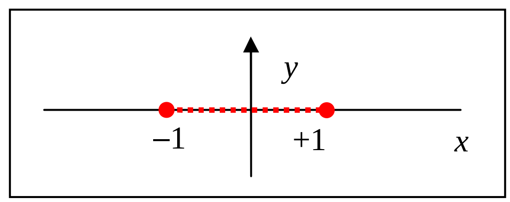
It is easy to find a branch of this expression that is continuous at t=0 (take for instance the principal branch of logarithm of (t+1)/(t–1))

We can find ***a* branch** of this multi-valued function using any of the three methods we learned in class:

METHOD 1: Choose $\frac{1}{2}\log\frac{1+z}{1-z} = \frac{1}{2}\left[\log(1+z) - \underbrace{\log(1-z)}_{i\pi+\log(z-1)}\right] \Rightarrow \frac{1}{2}\left[\log_p(z+1) - \log_p(z-1) - i\pi\right]$

$\log_p(z+1)$ has a brunch cut (jump by $2i\pi$) along $(-1, \infty)$

$\log_p(z-1)$ has a brunch cut (jump by $2i\pi$) along $(+1, \infty)$



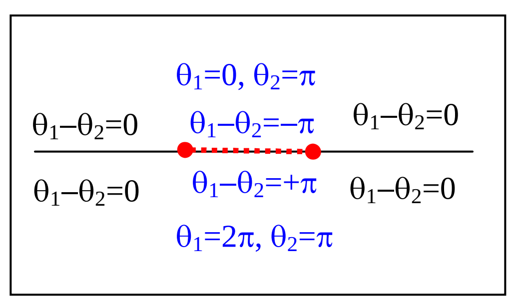
Subtracting the two cancels the jump everywhere except the cut (–1,+1), where the function jumps by $i\pi$.

METHOD 2 (mapping) Choose $\frac{1}{2}\log\frac{1+z}{1-z} = \frac{1}{2}\log_p \underbrace{\frac{1+z}{1-z}}_{x}$

Principal branch has a cut when $x>0$. Solving for z yields $z = \frac{x-1}{x+1}$. As x varies from 0 to infinity, z covers real interval (–1,+1). Function jumps by $i\pi$ as this branch is crossed from above.

METHOD 3 (Independent parametrization of factors)

$$\frac{1}{2}\log\frac{1+z}{1-z} = \frac{1}{2}\log\frac{\overbrace{1+z}^{r_1\exp(i\theta_1)}}{\underbrace{1-z}_{-r_2\exp(i\theta_2)}} = \frac{1}{2}\ln\frac{r_1}{r_2} + i\frac{\pi+\theta_1-\theta_2}{2}, \quad \underbrace{0<\theta_1,\theta_2<2\pi}_{\text{BRANCH DEFINITION}}$$



Function jumps by $i\pi$ as the branch is crossed from above

5) (16pts) Use parametrization to show that the following integral is zero over any circle around the origin:

$$\oint_{|z|=R}\left(\frac{1}{z}+z\right)dz=\int_0^{2\pi}\left(\frac{1}{\mathrm{R}\,e^{-it}}+\mathrm{R}\,e^{it}\right)\mathrm{R}\,ie^{it}\,dt=\int_0^{2\pi}\left(\frac{1}{\mathrm{R}}e^{it}+\mathrm{R}\,e^{it}\right)\mathrm{R}\,ie^{it}\,dt=i\left(1+\mathrm{R}^2\right)\int_0^{2\pi}e^{2it}\,dt=\frac{1+\mathrm{R}^2}{2}e^{2it}\Big|_0^{2\pi}=0$$

Does it follow that the integrand has an anti-derivative everywhere in the domain C/{0}?
Does it follow that the integrand is analytic in this domain? Explain your answers.

No, neither of the above statements is true in this case, since the integral may well be non-zero along a non-circular Jordan contour (one can try for instance a rectangular closed path)

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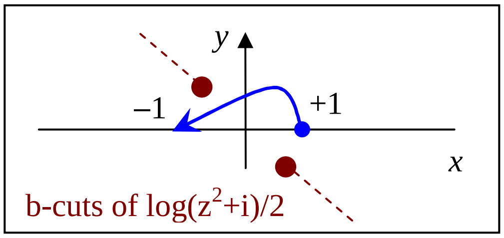
6) (16pts) Without resorting to parametrization, calculate $\int_C \frac{z\,dz}{z^2+i}$ along two different contours:

a) C = any contour from z = 1 to z = –1 not containing singularities of integrand

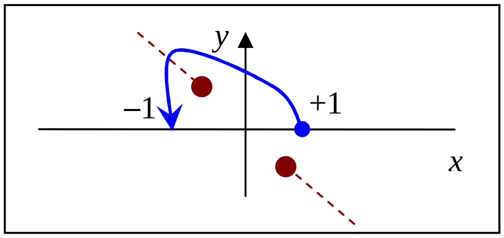
First, note that the integrand has two singularities at $z_{1,2}=(-i)^{1/2}=\pm(e^{-i\pi/2})^{1/2}=\pm e^{-i\pi/4}=\pm\frac{1-i}{\sqrt{2}}$



This integral *does* depend on the contour choice. If there are no singularities between the contour and the real axis (see figure), we can deform the contour to the interval of the real axis, along which the integral is zero since $x/(x^2+i)$ is an odd function.

We can also use the Fundamental Theorem of Calculus; **if the contour does not intersect the branch cut of the anti-derivative log(z²+i)/2** (you had to make this comment for full credit) we obtain:

$$\int_C \frac{z\,dz}{z^2+i}=\frac{1}{2}\log(z^2+i)\Big|_1^{-1}=\frac{1}{2}\log(1+i)-\frac{1}{2}\log(1+i)=\boxed{0}$$



Optional: this is not true if there is a pole between the contour and the real axis (see bottom figure). In this case we can close the contour along the real axis (which does not contribute to the integral according to above), and use the Cauchy Integral Formula:

$$\int_C \frac{z\,dz}{z^2+i}=\oint \frac{z\,dz}{z^2+i}=2\pi i\frac{z_1}{z_1-z_2}=\frac{2\pi i}{1-\underbrace{z_2/z_1}_{-1}}=\boxed{i\pi}$$

Note that this equals the jump of the anti-derivative across its branch cut (maroon dashed line)

b) C = circle of radius 2 around the origin in the positive direction

Use the Cauchy Integral Formula:

$$\int_C \frac{z\,dz}{z^2+i}=\int_C \frac{z\,dz}{(z-z_1)(z-z_2)}=2\pi i\left(\frac{z_1}{z_1-z_2}+\frac{z_2}{z_2-z_1}\right)=\boxed{2\pi i}$$

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7) (15pts) Parametrize the integral $\oint\limits_{|z|=1} \frac{e^{\alpha z}}{z} dz$ (where α is a real constant) and use the Cauchy Integral Formula to show that $\int_0^{\pi} e^{\alpha \cos\theta} \cos(\alpha \sin\theta) d\theta = \pi$.

By Cauchy Integral Formula we have

$$\oint\limits_{|z|=1} \frac{e^{\alpha z}}{z} dz = 2i\pi e^{\alpha \cdot 0} = 2i\pi$$

Also using parametrization $z=e^{it}$, we obtain

$$\int_0^{2\pi} \frac{e^{\alpha \exp(i\theta)}}{e^{i\theta}} ie^{i\theta} d\theta = i\int_0^{2\pi} e^{\alpha \exp(i\theta)} d\theta = i\int_0^{2\pi} e^{\alpha \cos\theta + i\alpha\sin\theta} d\theta = i\int_0^{2\pi} e^{\alpha \cos\theta} e^{i\alpha\sin\theta} d\theta$$

$$= i\int_0^{2\pi} e^{\alpha\cos\theta} \{\cos(\alpha\sin\theta) + i\sin(\alpha\sin\theta)\} d\theta = -\int_0^{2\pi} e^{\alpha\cos\theta} \sin(\alpha\sin\theta) d\theta + i\int_0^{2\pi} e^{\alpha\cos\theta}\cos(\alpha\sin\theta) d\theta$$

Now, the real part of this integral is zero (it's an odd function), while the imaginary part has to equal $2i\pi$ by the Cauchy Integral Formula, which gives us the final result

$$\boxed{\int_0^{2\pi} e^{\alpha\cos\theta}\cos(\alpha\sin\theta) d\theta = 2\pi}$$

Since the integrand is even with respect to $\theta=\pi$, we can divide the integration interval by half, yielding the final answer of π